PLANE STEADY PROBLEM OF HEAT-CONDUCTION THEORY FOR A HYPERBOLIC CYLINDER WITH BOUNDARY CONDITIONS OF THE THIRD KIND

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Fredholm integral equations of the second kind are obtained for the temperature distribution at the surface of a hyperbolic cylinder which is cooled in accordance with Newton's law. The integral equations permit solution by the successive-approximation method at small values of the Biot number.

It is known that boundary conditions of the third kind do not permit effective use of the conformal-mapping method in solving plane potential-theory problems. However, by the additional use of accurate solutions for wedge-shaped bodies [1], certain regions of shape asymptotically close to a wedge may be considered. In particular, conformal mapping of a wedge into a hyperbolic cylinder allows Fredholm integral equations of the second kind to be obtained for the temperature distribution at the cylinder surface.

Formulation of the Problem

Suppose that it is required to find the Laplace equation

$$\Delta u = 0, \quad (x, \ y) \in \Omega \tag{1}$$

with the boundary condition

$$\frac{\partial u}{\partial n} + hu\Big|_{\Gamma} = \varphi(p), \quad p \in \Gamma, \tag{2}$$

where

$$\Omega + \Gamma = \{(x, \ y) | \, x \geqslant \sqrt{y^2 + c^2 \sin^2 \gamma} \ \mathrm{ctg}\gamma, \ -\infty < y < +\infty\};$$
$$0 < \gamma < \pi,$$

$\partial u/\partial n$ is the derivative along the direction of the external normal; h, a positive constant; 2c, distance between foci of the hyperbola; 2γ, angle between asymptotes of the hyperbola; $\varphi(p) \in L_2(T)$, a given function.

To solve Eqs. (1) and (2), a curvilinear coordinate system $(\rho, \ \theta)$ is introduced, related to the Cartesian coordinates (x, y) as follows [2]:

$$z = \frac{c}{2}\left[(\xi^m + \sqrt{\xi^{2m} - 1})^{\frac{1}{m}} + (\xi^m - \sqrt{\xi^{2m} - 1})^{\frac{1}{m}}\right], \tag{3}$$

where

$$z = x + iy, \quad m = \frac{\pi}{2\gamma}, \quad \xi = \rho \exp i\theta, \ 0 \leqslant \rho < \infty, \quad -\gamma \leqslant \theta \leqslant +\gamma.$$

The function (3) performs the mutually single-valued mapping of a wedge with vertex angle 2γ onto a hyperbolic cylinder; $0 < \gamma < \pi/2$ corresponds to the interior region of the right-hand branch of the hyperbola and $\pi/2 < \gamma < \pi$ to the exterior region of the left-hand branch of the hyperbola. For $\gamma = \pi/2$ and $\gamma = \pi$, the mapping function is linear. Following the analytic-continuation principle for a harmonic function [2], the solution of the problem will be sought in the form of a sum

$$u = u^+ + u^-, \tag{4}$$

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where

$$u^+(\rho,\ \theta)=u^+(\rho,\ -\theta);\quad u^-(\rho,\ \theta)=-u^-(\rho,\ -\theta).$$

The functions $u^{(\pm)}(\rho,\ \theta)$ satisfy the equation*

$$\rho\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right)+\frac{\partial^2 u}{\partial\theta^2}=0,\quad \begin{array}{l}0<\rho<\infty,\\ 0<\theta<\gamma,\end{array} \tag{5}$$

with the boundary condition

$$\frac{1}{\rho}\frac{\partial u}{\partial\theta}+ha(\rho)\,u\Big|_{\theta=\gamma}=a(\rho)\,f(\rho),\quad 0<\rho<\infty \tag{6}$$

and the conditions on the symmetry lines

$$\begin{array}{l}\dfrac{\partial u^+}{\partial\theta}=0,\quad \theta=0,\quad 0<\rho<\infty,\\ u^-=0,\quad \theta=0,\quad 0<\rho<\infty,\end{array} \tag{7}$$

where

$$f^+(\rho)=\frac{\varphi(\rho,\ \gamma)+\varphi(\rho,\ -\gamma)}{2},\quad f^-(\rho)=\frac{\varphi(\rho,\ \gamma)-\varphi(\rho,\ -\gamma)}{2};$$

$$a(\rho)=\left|\frac{dz}{d\xi}\right|=\frac{c\rho^{m-1}}{2\sqrt{\rho^{2m}+1}}\sqrt{(\sqrt{\rho^{2m}+1}+\rho^m)^{\frac{2}{m}}+(\sqrt{\rho^{2m}+1}-\rho^m)^{\frac{2}{m}}-2\cos\frac{\pi}{m}}, \tag{8}$$

$$\theta=\gamma,\quad 0<\rho<\infty,\quad m>\frac{1}{2}.$$

Reducing the Problem to the Solution of Integral Equations

The significant feature of the mapping (3) which allows the Fredholm integral equations of the second kind to be obtained is that it is possible to write the modulus of the derivative of the mapping function (8) on the boundary $\theta=\gamma$ in the form

$$a(\rho)=2^{\frac{1-m}{m}}c\,[1+b(\rho)], \tag{9}$$

where

$$\int_0^\infty|b(\rho)|\,d\rho<\infty,\quad 0<\gamma<\pi.$$

Substituting Eq. (9) into Eq. (6) gives

$$\frac{1}{\rho}\frac{\partial u}{\partial\theta}+Hu\Big|_{\theta=\gamma}=a(\rho)\,f(\rho)-Hb(\rho)\,u=F(\rho),\quad 0<\rho<\infty, \tag{10}$$

where $H=2^{\frac{1-m}{m}}hc$ is the Biot number.

Assuming that $F(\rho)$ is a known function, the solution of Eqs. (5), (7), and (10) is written in the form [1]

$$u(\rho,\ \theta)=\int_0^\infty\frac{\psi_\gamma(\lambda\rho,\ \theta)}{\lambda+H}d\lambda\int_0^\infty F(r)\,\psi_\gamma(\lambda r)\,dr. \tag{11}$$

Setting $\theta=\gamma$ in Eq. (11), and substituting into Eq. (10), integral equations for the temperature distribution at the hyperbolic-cylinder surface are obtained:

$$u(\rho)+H\int_0^\infty u(r)\,b(r)\,K_\gamma(H,\ r,\ \rho)\,dr=\int_0^\infty a(r)\,f(r)\,K_\gamma(H,\ r,\ \rho)\,dr,\quad 0\leqslant\rho<\infty, \tag{12}$$

*The symbol (±) will be omitted below.

where

$$K_\gamma(H, r, \rho) = \int_0^\infty \frac{\psi_\gamma(\lambda r)\psi_\gamma(\lambda\rho)\,d\lambda}{\lambda + H}.$$

It may be shown that the linear operator (12) satisfies the condition for the applicability of the compressed-mapping principle in the metric space $\{u(\rho)\}$ with the norm

$$\|u(\rho)\| = \left(\int_0^\infty |b(\rho)|\, u^2(\rho)\, d\rho\right)^{\frac{1}{2}} \tag{13}$$

and the distance

$$\delta(u, v) = \|u - v\|. \tag{14}$$

According to the compressed-mapping principle, the successive-approximation method may be used to solve Eq. (12) under the condition [3]

$$H^2 \int_0^\infty \int_0^\infty |b(r)||b(\rho)| K_\gamma^2(H, r, \rho)\, dr\, d\rho < 1. \tag{15}$$

The condition (15) is satisfied when H is sufficiently small. In addition, when m is an integer, the kernel (12) is expressed in final form in terms of an integral power function, and permits effective solution by numerical methods [4, 5].

Example

The problem is to find the steady temperature distribution at the surface of a hyperbolic cylinder if there is a linear heat source at the focus of the hyperbola and the surface is cooled by Newton's law in a medium of zero temperature.

The solution is sought in the form of a sum,

$$u = \frac{q}{4\pi K} \ln\left[\frac{\rho^{2m} + 2\rho^m \cos m\theta + 1}{\rho^{2m} - 2\rho^m \cos m\theta + 1}\right] + u^+, \tag{16}$$

where u^+ is the desired temperature; K is the thermal conductivity; q is the heat liberated by the source in unit time per unit length. The temperature distribution $u = u^+$ at the surface $\theta = \gamma$ is found from Eq. (12). At sufficiently small H, the solution of the equation is written in the form

$$u^+(\rho) = \int_0^\infty a(r) f^+(r) K_\gamma^+(H, r, \rho)\, dr\, [1 + O(H \ln H)],\ 0 \leqslant \rho < \infty, \tag{17}$$

where

$$a(r) f^+(r) = \frac{q}{\pi K} \frac{m r^{m-1}}{r^{2m} + 1}.$$

Numerical values of Eq. (17) are given in [4] for m = 1, 2, 3.

In conclusion, note that analogous results may be obtained in the case of a linear source q at an arbitrary position in the region Ω.

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